A Lecture Note

ON

Application of Congruence

prepared by

Dr. Soovoojeet Jana

Assistant Professor

Dept. of Mathematics

Ramsaday College

Amta-711401, Howrah, West Bengal

E-mail: soovoojeet@gmail.com

- ISBN stands for International Standard Book Number
- ISSN stands for International Standard Serial Number
- UPC stands for Universal Product Code
- In 1970, ISBN was introduced and till the year 2006, every ISBN was of 10 numerics.
- January, 2007 onwards, ISBN length has changed from 10 digits to 13 digits.

ISBN-10

An ISBN-10 is an expression of alphanumeric numbers containing 10 digits and therefore ISBN of any book having ISBN-10 can be expressed as $a_1a_2a_3a_4a_5a_6a_7a_8a_9c$

where each of $a_i (i = 1, 2, ..., 9)$ are numerical numbers containing either of 0, 1, 2, 3, ..., 9.

The last digit of ISBN, c (which is known as check digit) is an alphanumeric number may take from 0, 1, 2, ..., 10. If the value of the check digit is 10 then we would represent it by the alphabet X. Thus the possible value of the check digit (c) is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, X.

The rule to determine the check digit c is due to the application of congruence and it is as follows :

The check digit c of the above mentioned ISBN should follow the following congruences :

$$1 \cdot a_1 + 2 \cdot a_2 + 3 \cdot a_3 + 4 \cdot a_4 + 5 \cdot a_5 + 6 \cdot a_6 + 7 \cdot a_7 + 8 \cdot a_8 + 9 \cdot a_9 + 10 \cdot c \equiv 0 \pmod{11}$$

$$i.e., \sum_{i=1}^{9} i \cdot a_i + 11c - c \equiv 0 \pmod{11} \quad i.e., \sum_{i=1}^{9} i \cdot a_i \equiv c \pmod{11} \qquad \ldots (1)$$

Determine whether the following ISBN is valid : 81-203-1147-7.

Solution Here the given ISBN is 81-203-1147-7.

Since the ISBN has 10 digits, therefore the ISBN is of ISBN–10 type and the check digit (*i.e.*, the last digit) 7 should follow the following congruence :

$$\sum_{i=1}^{9} i \times a_i \equiv 7 \pmod{11} \qquad \ldots (1)$$

where a_i is the digit of the ISBN at i-th position ($i = 1, 2, \ldots, 9$).

Now we have, $\sum_{i=1}^{9} i \times a_i = 8 \times 1 + 1 \times 2 + 2 \times 3 + 0 \times 4 + 3 \times 5 + 1 \times 6 + 1 \times 7 + 4 \times 8 + 7 \times 9$

$$= 8 + 2 + 6 + 0 + 15 + 6 + 7 + 32 + 63 = 139 \equiv 7 \pmod{11}$$

Thus the check digit should be 7 and hence the given ISBN is valid.

Determine whether the ISBN 3-540-19102-X is valid.

Solution Here the given ISBN is 3-540-19102-X.

Since it has 10 digits, therefore the ISBN is of ISBN-10 type and the check digit (*i.e.*, the last digit) × (equivalent to 10 in decimal system) should follow the congruence given below :

$$\sum_{i=1}^{9} i \times a_i \equiv 10 \pmod{11} \qquad \ldots (1)$$

where a_i is the digit of ISBN at i-th position (i = 1, 2, ..., 9).

Now, we have $\sum_{i=1}^{9} i \times a_i = 3 \times 1 + 5 \times 2 + 4 \times 3 + 0 \times 4 + 1 \times 5 + 9 \times 6 + 1 \times 7 + 0 \times 8 + 2 \times 9$

$= 3 + 10 + 12 + 0 + 5 + 54 + 7 + 0 + 18 = 109 \equiv 10 \pmod{11}$.

Thus the check digit of the given ISBN should be X and hence the given ISBN is a valid ISBN.

Determine the following ISBN is valid or not 0-27-040035-5.

<u>Solution</u> Here the given ISBN is 0-27-040035-5 and since it has 10 digits therefore it is of ISBN–10 type with check digit 5.

Now the validity of that ISBN can be checked by validating the following congruence :

$$\sum_{i=1}^{9} i \times a_i \equiv 5 \pmod{11} \qquad \text{... (1)}$$

where a_i is the digit of the ISBN at i-th position. Therefore for the given ISBN, we have

$$\sum_{i=1}^{9} i \times a_i = 0 \times 1 + 2 \times 2 + 7 \times 3 + 0 \times 4 + 4 \times 5 + 0 \times 6 + 0 \times 7 + 3 \times 8 + 5 \times 9$$

$$= 0 + 4 + 21 + 0 + 20 + 0 + 0 + 24 + 45 = 114 \equiv 4 \pmod{11}.$$

Thus in order to make the above ISBN, a valid ISBN, the check digit should be 4 but it is provided as 5. Hence the ISBN 0-27-040035-5 is not valid ISBN.

ISSN

A valid ISSN of journal contains eight digits where the last digit is the check digit, which may be any digit from 0,1,2,……,9 or the alphabet X representing 10. The formula to determine the check digit c is given by:

$$c + \sum_{i=1}^{7}(i+1) \times a_{8-i} \equiv 0 \pmod{11} \qquad (3)$$

Illustration The International Standard Serial Number (ISSN) of an international journal 'International Journal of Dyamics and Control' published by Springer is 2195-268X. Here the check digit is the last digit X (equivalent to 10). Now let us check whether the check digit is correct or not. According to the formula given in (3), the check digit, c, say can be obtained from the following formula :

$c + 8 \times 2 + 7 \times 1 + 6 \times 9 + 5 \times 5 + 4 \times 2 + 3 \times 6 + 2 \times 8 \equiv 0 \pmod{11}$

i.e., $c + 16 + 7 + 54 + 25 + 8 + 18 + 16 \equiv 0 \pmod{11}$

i.e., $c + 144 \equiv 0 \pmod{11}$ *i.e.*, $c + 1 \equiv 0 \pmod{11}$, *i.e.*, $c = 10$.

UPC

UPC (Universal Product Code) are used in USA and Canada and it is 12 digits where as EAN (European Article Number) codes are of 13 digit and they are used in rest of the countries. If you get 12 digit UPC codes but you want to use it as EAN codes, you can do so by adding zero(0) in front of the codes.

However, here by UPC we will mean UPC-A and the details of **UPC-A** is discussed next slide.

UPC-A

The UPC-A check digit may be calculated as follows:

- Sum the digits at odd-numbered positions (first, third, fifth,..., eleventh).
- Multiply the result by 3.
- Add the digit sum at even-numbered positions (second, fourth, sixth,..., tenth) to the result.
- Find the result modulo 10 (i.e. the remainder, when divided by 10) and call it r.
- If r is zero, then the check digit is 0; otherwise the check digit is $10 - r$.

Example

For example, in a UPC-A barcode "03600028155c", where *c* is the unknown check digit, *c* may be calculated by:

- Sum the odd-numbered digits (0 + 6 + 0 + 2 + 1 + 5 = 14).
- Multiply the result by 3 (14 × 3 = 42).
- Add the even-numbered digits (42 + (3 + 0 + 0 + 8 + 5) = 58).
- Find the result modulo 10 (58 (mod 10) = 8(=r)).
- Since $\mathrm{r} \neq 0$, subtract r from 10 (10 − r = 10 − 8 = 2).
- Thus, the check digit c is 2.

UPC in other countries

The formula for finding check digit of UPC containing 13 digit or EAN-13 is as follows:

$$\sum_{i=1,\, i \text{ is odd}}^{12} a_i + 3 \sum_{i=2,\, i \text{ is even}}^{12} a_i + c \equiv 0 \pmod{10} \qquad (4)$$

Ex. Check whether the UPC 8906032713739 is a valid UPC or not

UPC is a valid or not, we first simplify the expression given at LHS of (4),

which is $\sum_{i=1,\, i \text{ is odd}}^{12} a_i + 3 \sum_{i=2,\, i \text{ is even}}^{12} a_i$. Thus for the present UPC,

We have $\sum_{i=1,\, i \text{ is odd}}^{12} a_i + 3 \sum_{i=2,\, i \text{ is even}}^{12} a_i = (8 + 0 + 0 + 2 + 1 + 7) + 3(9 + 6 + 3 + 7 + 3 + 3)$

$= 18 + 3 \times 31 = 111 \equiv 1 \pmod{10}$

Therefore by the formula stated in (4) [or equivalently in (5)], the check digit c should be $10 - 1 = 9$. Hence the given UPC 8906032713739 is a valid UPC.

Find the check digit in the following ISBN : 978-81-8128-639-?

To determine c, the check digit, we first make the following calculation :

$$\sum_{i=1,\, i \text{ is odd}}^{12} a_i + 3 \sum_{i=2,\, i \text{ is even}}^{12} a_i \text{ , where } a_i \text{ is the } i\text{-th digit of the ISBN}$$

$$= 9 + 8 + 1 + 1 + 8 + 3 + 3\,(7 + 8 + 8 + 2 + 6 + 9) = 30 + 120 = 150$$

Now, we have $150 \equiv 0 \pmod{10}$ and hence the check digit should be 0.

Thus the correct ISBN is 978-81-8128-639-0.

Note: ISBN-13 has same rule as of UPC which is discussed in the formula (4).

Formation of Round Robin Tournament

In this case the problem is to formulate a schedule of a league tournament participating $n \in N$ teams. The rule should be in each round no team should be playing more than once and in the whole tournament all the teams should be playing with all the rest teams exactly once. Using congruence, the formula to determine which team will be playing to exactly which team in a particular round is given as below:

$$\text{Formula:} \quad t_i + t_j = r \ (mod\ n)$$

Here, n represents number of total teams in the tournament.

r represnt the particular round of play

and t_i and t_j represent i th team and j th team where both $1 \leq i, j \leq 8$.

We provide two examples two formulate the round robin tournaments consisting of four, five and six teams respectively.

A Round Robin Tournament time table having four teams is as follows.

Team → / Rounds ↓	t_1	t_2	t_3	t_4
1	t_4	t_3	t_2	t_1
2	bye	t_4	bye	t_1
3	t_2	t_1	t_4	t_3
4	t_3	bye	t_1	bye

A Round Robin Tournament for five teams is as follows:

Team → / Rounds ↓	t_1	t_2	t_3	t_4	t_5
1	t_5	t_4	bye	t_2	t_1
2	bye	t_5	t_4	t_3	t_2
3	t_2	t_1	t_5	bye	t_3
4	t_3	bye	t_1	t_5	t_4
5	t_4	t_3	t_2	t_1	bye

A Round Robin Tournament for six teams is as follows:

Team → / Rounds ↓	1	2	3	4	5	6
1	6	5	4	3	2	1
2	bye	6	5	bye	3	2
3	2	1	6	5	4	3
4	3	bye	1	6	bye	4
5	4	3	2	1	6	5
6	5	4	bye	2	1	bye

SUGGESTED READING

Fundamental Mathematics (SEM-II), B. Pal, S. Raychaudhuri and S. Jana, Santra Publishers.

Send your feedback on soovoojeet@gmail.com /mathsdeptrsc@gmail.com .