2SH CC-4

RAMSADAY COLLEGE

Homomorphism(1)

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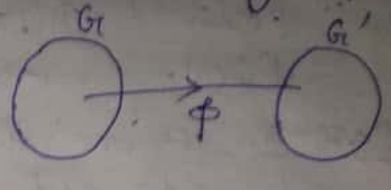
- **Definition:-** Let $(G,\circ)$ and $(G',*)$ be two groups. A mapping $\phi: G \to G'$ is said to be a homomorphism if $\phi(a\circ b) = \phi(a) * \phi(b)$, $\forall\, a,b \in G$.



- **Definitions:-**
  - A one-to-one homomorphism is said to be a monomorphism.
  - An onto homomorphism is said to be an epimorphism.
  - A homomorphism is said to be an isomorphism if it is both a monomorphism and an epimorphism.
  - An isomorphism from a group onto itself is said to be an automorphism.

Example-1:- Let $G=(\mathbb{Z},+)$, $G'=(2\mathbb{Z},+)$ and a mapping $\phi: G\to G'$ be defined by $\phi(a)=2a$, $a\in G$. Examine if ϕ is a homomorphism.

Ans: Let $a,b\in G$. Then $a+b\in G$ and
$\phi(a)=2a$, $\phi(b)=2b$, $\phi(a+b)=2(a+b)$

$\therefore\ \phi(a+b)=2(a+b)=2a+2b=\phi(a)+\phi(b)$.

$\therefore\ \phi(a+b)=\phi(a)+\phi(b)$, $\forall\, a,b\in G$.

This shows that ϕ is a homomorphism.

Example-2:- Let $G=S_3$, $G'=(\{1,-1\},\cdot)$ and let $\phi: G\to G'$ be defined by

$$\phi(\alpha) = 1, \text{ if } \alpha \text{ be an even permutation in } S_3$$
$$= -1, \text{ if } \alpha \text{ be an odd permutation in } S_3.$$

Examine if ϕ is a homomorphism.

Ans:- Let $\alpha,\beta\in S_3$. Then $\alpha\beta\in S_3$.

Case-1. Let α,β be both even permutations. Then $\alpha\beta$ is even.

$\therefore\ \phi(\alpha)=1$, $\phi(\beta)=1$ and $\phi(\alpha\beta)=1$.

$\therefore\ \phi(\alpha\beta)=1=\phi(\alpha)\cdot\phi(\beta)$.

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Case-2. Let α, β be both odd. Then $\alpha\beta$ is even.

and $\phi(\alpha\beta) = 1$.

$\therefore \phi(\alpha) = -1, \phi(\beta) = -1$

$\therefore \phi(\alpha\beta) = 1 = (-1)\cdot(-1) = \phi(\alpha)\cdot\phi(\beta)$

Case-3. Let one of α, β be odd and the other be even.

Let α be odd, β be even. Then $\alpha\beta$ is odd.

$\therefore \phi(\alpha) = -1, \phi(\beta) = 1$ and $\phi(\alpha\beta) = -1$

$\phi(\alpha\beta) = -1 = \phi(\alpha)\cdot\phi(\beta)$.

Hence $\phi(\alpha\beta) = \phi(\alpha)\cdot\phi(\beta), \forall \alpha, \beta \in S_3$.

This proves that ϕ is a homomorphism.

Here ϕ is not injective but surjective. Therefore ϕ is an epimorphism but not a monomorphism.

Theorem-1:- Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi: G \to G'$ be a homomorphism. Then

(i) $\phi(e_G) = e_{G'}$;

(ii) $\phi(a^{-1}) = \{\phi(a)\}^{-1}, \forall a \in G$

(iii) if $a \in G$, then $\phi(a^n) = \{\phi(a)\}^n$, n being an integer.

(iv) if $a \in G$ and $o(a)$ is finite then $o(\phi(a))$ is a divisor of $o(a)$.

** **Definition**:- Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi: G \to G'$ be a homomorphism. The image of ϕ, denoted by $\mathrm{Im}\,\phi$ is a subset of G' defined by $\mathrm{Im}\,\phi = \{\phi(a) : a \in G\}$.

$\mathrm{Im}\,\phi$ is also called the homomorphic image of ϕ and is denoted by $\phi(G)$.

Theorem-2:- Let $(G, \circ)$ and $(G', *)$ be two groups and $\phi: G \to G'$ be a homomorphism. Then $\phi(G)$ is a subgroup of G'.

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Theorem-3:- Let $(G,\circ)$ and $(G',*)$ be two groups and let $\phi: G \to G'$ be an epimorphism. Then

(i) if G be commutative then G' is commutative, but the converse is not true.

(ii) if G be cyclic then G' is cyclic, but the converse is not true.

Proof $\mapsto$

(i) Let a', b' be two elements in G'.

Since ϕ is an onto homomorphism, $\exists$ elements a, b in G s.t. $\phi(a) = a'$, $\phi(b) = b'$.

$$a' * b' = \phi(a) * \phi(b) = \phi(a \circ b), \text{ since } \phi \text{ is a homomorphism}$$
$$= \phi(b \circ a), \text{ since } G \text{ is commutative}$$
$$= \phi(b) * \phi(a)$$
$$= b' * a'$$

$\therefore a' * b' = b' * a'$, $\forall a', b' \in G'$.

So, G' is a commutative group.

• The converse is not true, follows from example-2.

(ii) Let $G = \langle a \rangle$. Then every element of G is of the form a^r, where $r \in \mathbb{Z}$.

Let $b' \in G'$. Since ϕ is an onto homomorphism, $\exists$ an element b in G s.t. $\phi(b) = b'$.

Since $b \in G$, $b = a^m$ for some integer m.

$$b' = \phi(b) = \phi(a^m) = \{\phi(a)\}^m$$

This shows that, every element in G' is of the form $\{\phi(a)\}^r$, where $\phi(a) \in G'$ and r is an integer.

Therefore, G' is cyclic and $G' = \langle \phi(a) \rangle$.

• The converse is not true, follows from example-2.

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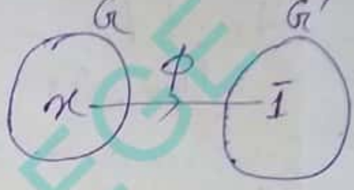
Note:- If a be a generator of the cyclic group G, then $\phi(a)$ is a generator of the cyclic group $\phi(G)$.

Example-3:- Show that there does not exist an onto homomorphism from the group S_3 to the group $(\mathbb{Z}_6, +)$.

Ans. Let, $G = S_3$, $G' = (\mathbb{Z}_6, +)$.

If possible let, $\phi : G \to G'$ be an onto homomorphism.

Now, $\bar{1} \in G'$ and $o(\bar{1}) = 6$.



Since ϕ is onto, so $\exists$ a pre-image of $\bar{1}$ in G.

Let x be the pre-image of $\bar{1}$ in G.

So, $\phi(x) = \bar{1} \Rightarrow o(\phi(x)) = o(\bar{1}) = 6$.

Since $x \in G$, then $o(x)$ is either 1 or 2 or 3.

Since ϕ is a homomorphism and $o(x)$ is finite, then $o(\phi(x)) \mid o(x) \Rightarrow 6 \mid o(x)$, which is an impossibility. So, ϕ does not exist.

Example-4:- Find all homomorphisms from the group $(\mathbb{Z}_8, +)$ to the group $(\mathbb{Z}_6, +)$.

Ans:- Let $G = (\mathbb{Z}_8, +)$, $G' = (\mathbb{Z}_6, +)$.

~~Let G~~ Then G is a cyclic group with $\bar{1}$ as a generator.

Let $\phi : G \to G'$ be a homomorphism.

Then $\phi(\bar{1})$ is a generator of $\phi(G)$.

Let, $\bar{a} \in G$.

Then $\phi(\bar{a}) = \phi(\bar{1} + \bar{1} + \cdots + \bar{1})$

$= a\,\phi(\bar{1})$, since ϕ is a homomorphism.

Now, ϕ is completely determined if $\phi(\bar{1})$ be known.

Now, $\bar{1} \in G \Rightarrow o(\bar{1}) = 8$.

Since ϕ is a homomorphism and $o(\bar{1})$ is finite, then $o(\phi(\bar{1})) \mid o(\bar{1})$ i.e., $o(\phi(\bar{1})) \mid 8$.

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Again, $\phi(\bar{1}) \in G' \Rightarrow o(\phi(\bar{1})) | o(G')$

$\Rightarrow o(\phi(\bar{1})) | 6$.

Therefore $o(\phi(\bar{1}))$ is a common divisor of 6 and 8.

So, $o(\phi(\bar{1}))$ is either 1 or 2.

So, $\phi(\bar{1}) = \bar{0}$ or $\bar{3}$ in $\mathbb{Z}_6$.

If, $\phi(\bar{1}) = \bar{0}$, then $\phi(\bar{a}) = \bar{0}, \forall a \in G \rightarrow (1)$

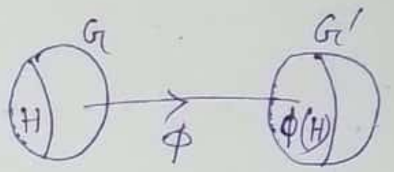
If $\phi(\bar{1}) = \bar{3}$, then $\phi(\bar{a}) = a\bar{3}, \forall a \in G \rightarrow (2)$.

In this case, $\phi(\bar{0}) = \bar{0}$, $\phi(\bar{1}) = \bar{3}$, $\phi(\bar{2}) = \bar{0}$, $\phi(\bar{3}) = \bar{3}$, $\phi(\bar{4}) = \bar{0}$, $\phi(\bar{5}) = \bar{3}$, $\phi(\bar{6}) = \bar{0}$, $\phi(\bar{7}) = \bar{3}$.

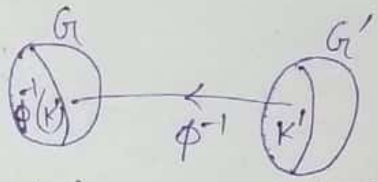
Therefore there are only two homomorphisms given by (1) and (2).

<u>Theorem-4</u> $\mapsto$ Let $(G, \circ)$ and $(G', *)$ be two groups and let $\phi: G \rightarrow G'$ be a homomorphism. Then

(i) if H be a subgroup of G, $\phi(H)$ is a subgroup of G';



(ii) if K' be a subgroup of G', $\phi^{-1}(K')$ is a subgroup of G.



[where $\phi^{-1}(K')$ is the inverse image of K' under the mapping ϕ. ϕ^{-1} may not be a mapping].

<u>Theorem-5</u> :- Let $(G, \circ)$ and $(G', *)$ be two groups and let $\phi: G \rightarrow G'$ be an onto homomorphism. Then

(i) if H be a <u>normal</u> subgroup of G, $\phi(H)$ is a <u>normal</u> subgroup of G';

(ii) if K' be a <u>normal</u> subgroup of G', $\phi^{-1}(K')$ is a <u>normal</u> subgroup of G, where $\phi^{-1}(K')$ is the inverse image of K' under the mapping ϕ.

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