Infinite Series :- (1)

● If $\{u_n\}$ is a sequence of real numbers, then the expression

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

i.e. the sum of the terms of the sequence, is called an infinite series.

The infinite series $u_1 + u_2 + \cdots + u_n + \cdots$ is usually denoted by $\sum_{n=1}^{\infty} u_n$ or simply by Σu_n.

If all the terms of the series Σu_n are positive. i.e. if $u_n > 0 \ \forall n$; then the series is called a series of positive terms.

A series in which the terms are alternatively +ve and −ve is called an alternating series.

Thus the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \cdots + \cdots (-1)^{n-1} u_n + \cdots$$

where $u_n > 0 \ \forall n$

is an alternating series.

● Partial sums :-

If $\Sigma u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is an infinite series where the terms may be +ve or −ve, then $S_n = u_1 + u_2 + \cdots + u_n$, the sum of its first n terms, is called the partial sum.

$\{S_n\}$ is a sequence called the sequence of partial sum of the infinite series Σu_n.

Then $\lim_{n\to\infty} S_n$ is said to be the sum of the series.

● Behaviour of an infinite series :-

An infinite series Σu_n converges, diverges or oscillates (finite or infinitely) according as the sequence $\{S_n\}$ of its partial sums converges, diverges or oscillate (finite or infinitely).

i) The series Σu_n converges if $\lim_{n\to\infty} S_n$ = finite value

ii) The series Σu_n diverges if $\lim_{n\to\infty} S_n = +\infty$ or $-\infty$

iii) The series Σu_n oscillates finitely if $\{S_n\}$ is bounded and neither converges nor diverges and oscillates infinitely if $\{s_n\}$ is unbounded and neither converges nor diverges.

• Some important series :-

1. Geometric Series :-

The infinite geometric series

$$1+r+r^2+r^3+r^4+\cdots+r^n+\cdots$$

i) converges for $r<1$

and ii) diverges to $+\infty$ for $r\geq 1$.

2. P-Series :-

A positive term infinite series $\sum \frac{1}{n^p}$;

i) converges if $p>1$

and ii) diverges if $p\leq 1$.

3). Harmonic Series :-

A series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$ is always diverges.

• Note: 1. A necessary condition for convergence of an infinite series $\sum u_n$ is the $\lim_{n\to\infty} u_n = 0$.

2. If $\sum u_n$ converges to u and c is a constant then $\sum c u_n$ converges to cu.

3. If $\sum u_n$ and $\sum v_n$ converge to u and v, then $\sum (u_n \pm v_n)$ converges to $(u \pm v)$.

• Comparison test :- [First type]

In this section, to test a given series whether it converges or not we compare the given series with a suitable selected series with known behaviour. The geometric and p-series are most frequently used for such purpose.

Test - I

If $\sum u_n$ and $\sum v_n$ be two +ve term series and $k \neq 0$, a fixed +ve real number (independent of n) and $\exists$ a +ve integer m such that $u_n \leq k v_n \ \forall n \geq m$, then

(i) $\sum u_n$ is convergent, if $\sum v_n$ is convergent.

and (ii) $\sum v_n$ is divergent, if $\sum u_n$ is divergent.

Test II :-
If Σu_n and Σv_n be two positive term series such that $\lim_{n\to\infty} \left(\frac{u_n}{v_n}\right) = l$. where l is a non-zero finite number, then two series converges or diverges together.

Test III :-
If Σu_n and Σv_n be two positive term series and $\exists$ a +ve integer m such that
$$\frac{u_n}{u_{n+1}} \geq \frac{v_n}{v_{n+1}} \quad \forall n \geq m.$$
If $l = 0$ then Σu_n is convergent if Σv_n is convergent. and also $l = \pm\infty$ then Σu_n is divergent if Σv_n is divergent.

then

(i) Σu_n is convergent, if Σv_n is convergent.

and (ii) Σv_n is divergent, if Σu_n is divergent.

Some important test for convergence of series :-

1. Cauchy's Root Test :-

Statement :- If Σu_n is a series of +ve terms such that $\lim_{n\to\infty} (u_n)^{1/n} = l$, then the series

(i) converges if $l < 1$,

(ii) diverges if $l > 1$,

and (iii) the test fails to give any definite information if $l = 1$.

Proof :- Case I : Let $l < 1$

Let us select a +ve no ϵ such that $l + \epsilon < 1$

Let, $l + \epsilon = \alpha < 1$.

Since $\lim_{n\to\infty} (u_n)^{1/n} = l$, so for any given $\epsilon > 0$, $\exists$ a +ve integer m such that $|(u_n)^{1/n} - l| < \epsilon \ \forall n \geq m$

or $l - \epsilon < (u_n)^{1/n} < l + \epsilon \ \forall n \geq m$

$$\Rightarrow (l-\epsilon)^n < u_n < (l+\epsilon)^n \quad \text{———} (1) \ \forall n \geq m.$$

Now (1) $\Rightarrow u_n < \alpha^n \ \forall n \geq m$.

But the series $\Sigma \alpha^n$ is a convergent geometric series (with common ratio $\alpha < 1$)

Thus by comparison test the series Σu_n is convergent.

Case II: Let $l > 1$
Let us select a +ve no ϵ such that $l - \epsilon > 1$
Let, $l - \epsilon = \beta > 1$
Now (i) gives
$\beta^n < u_n \ \forall n \geq m$

But the series $\sum \beta^n$ is divergent geometric series (with common ratio $\beta > 1$).
Thus by comparision test the series $\sum u_n$ is divergent.

Case III:-
Let $l = 1$
First we consider the series $\sum \frac{1}{n}$
$\therefore u_n = \frac{1}{n}$ and $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n}\right)^{1/n} = 1$
Next we consider the series $\sum \frac{1}{n^2}$
$\therefore u_n = \frac{1}{n^2}$, $\lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^2}\right)^{1/n}$
$= \lim_{n\to\infty} \left(\frac{1}{n^{1/n}}\right)^2 = 1$

Thus in both cases, $\lim_{n\to\infty} (u_n)^{1/n} = 1$, while $\sum \frac{1}{n}$ is divergent and $\sum \frac{1}{n^2}$ is convergent.
Therefore, for $l = 1$, the test does not give any definite information.

2. D'Alembert ratio test :-

Statement :-
If $\sum u_n$ is a positive term series, such that $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then the series. [C.H-94]
i) converges if $l < 1$.
(ii) diverges if $l > 1$.
and (iii) the test fails, if $l = 1$.

Proof :-
Case I :-
Let, $l < 1$.
Let us select a +ve no ϵ, such that $l + \epsilon < 1$.
Let $l + \epsilon = \alpha < 1$.

Since $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, so for any given $\epsilon > 0$, $\exists$ a +ve integer m such that

$$\left| \frac{u_{n+1}}{u_n} - l \right| < \epsilon \quad \forall\, n \geq m$$

i.e. $l - \epsilon < \frac{u_{n+1}}{u_n} < l + \epsilon \quad \forall\, n \geq m$ —— (1)

(1) gives $\frac{u_{n+1}}{u_n} < l + \epsilon = \alpha \quad \forall\, n \geq m$

For $n \geq m$,

$$\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdot \frac{u_{m+3}}{u_{m+2}} \cdots \frac{u_n}{u_{n-1}} < \alpha^{n-m}$$

$\Rightarrow u_n < u_m \alpha^{n-m}$

i.e. $u_n < \left(\frac{u_m}{\alpha^m}\right) \alpha^n \quad \forall\, n \geq m,\ \alpha < 1$

$\Rightarrow u_n < k \alpha^n$, $\quad k = \frac{u_m}{\alpha^m}$, a fixed no.

But $\Sigma \alpha^n$ is a convergent series (geometric) with common ratio $\alpha < 1$

Thus by comparison test Σu_n is convergent.

Case II :-

Let, $l > 1$

Let us select a +ve no ϵ such that $l - \epsilon > 1$.

Let, $l - \epsilon = \beta > 1$

From (1), we have

$$\beta = l - \epsilon < \frac{u_{n+1}}{u_n} \quad \forall\, n \geq m$$

i.e. $\frac{u_{n+1}}{u_n} > \beta \quad \forall\, n \geq m$.

For $n \geq m$.

$$\frac{u_n}{u_m} = \frac{u_{m+1}}{u_m} \cdot \frac{u_{m+2}}{u_{m+1}} \cdots \frac{u_n}{u_{n-1}} > \beta^{n-m}$$

i.e. $u_n > \frac{u_m}{\beta^m} \cdot \beta^n \quad \forall\, n \geq m$

or, $u_n > k' \beta^n \quad \forall\, n \geq m$,

where $k' = \frac{u_m}{\beta^m}$, a finite no

But $\sum p^n$ is a divergent geometric series with common ratio $p>1$.
Thus, by comparison test the series $\sum u_n$ is divergent.

case III: = let, $l=1$.
For the series $\sum \frac{1}{n}$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n}{n+1}$

$$= \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

and for the series $\sum \frac{1}{n^2}$, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$

$$= 1$$

Thus in both series $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = 1$, but the series $\sum \frac{1}{n}$ is divergent while $\sum \frac{1}{n^2}$ is convergent.
Therefore, for $l=1$, the test does not give any definite conclusion.

Q:1
Test the convergence of the following series:

a) $\sum_{n=1}^{\infty} \frac{1}{n^2+a^2}$

b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

c) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

d) $\sum_{n=1}^{\infty} \left(\sqrt{n^4+1}-\sqrt{n^4-1}\right)$.

e) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n^2}\right)$

f) $\sum_{n=1}^{\infty} \frac{\left(\sqrt{n+1}-\sqrt{n-1}\right)}{n}$

g) $\frac{1+2}{2^3} + \frac{1+2+3}{3^3} + \frac{1+2+3+4}{4^3} + \cdots$

h) $\frac{1.2}{3^2.4^2} + \frac{3.4}{5^2.6^2} + \frac{5.6}{7^2.8^2} + \cdots$

i) $\frac{1}{1.2.3} + \frac{3}{2.3.4} + \frac{5}{3.4.5} + \frac{7}{4.5.6} + \cdots$

j) $\frac{1}{\sqrt{1.2}} + \frac{1}{\sqrt{2.3}} + \frac{1}{\sqrt{3.4}} + \cdots$

k) $\sum_{n=2}^{\infty} \frac{\log n}{\sqrt{n+1}}$

l) $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} + \cdots$

m) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n!}}$ n) $\sum_{n=1}^{\infty} \sqrt[3]{n^3+1} - n$.

o) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \tan\frac{1}{n}$.

Solution :-

(a). Let, $u_n = \frac{1}{n^2+a^2}$ and take $v_n = \frac{1}{n^2}$

$\therefore \frac{u_n}{v_n} = \frac{\frac{1}{n^2+a^2}}{\frac{1}{n^2}} = \frac{n^2}{n^2+a^2} = \frac{1}{1+\frac{a^2}{n^2}}$

$\therefore \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{1}{1+\frac{a^2}{n^2}} = 1.$

Then. $\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{1}{n^2} = 0.$

Also the series $\sum v_n$ is convergent.

Thus by comparison test the given series $\sum_{n=1}^{\infty} u_n$ is convergent.

b) Let, $u_n = \frac{1}{\sqrt{n}+\sqrt{n+1}}$ and take $v_n = \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}}$

$\therefore \lim \frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n}+\sqrt{n+1}} = \frac{2}{1+1+\sqrt{\frac{1}{n}}}$ $= \cancel{1 \neq 0}$.

$\Rightarrow \lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{2}{1+1+\sqrt{\frac{1}{n}}} = 1. \neq 0$

Also. $\lim_{n\to\infty}(v_n) = \lim_{n\to\infty} \frac{1}{2\sqrt{n}} = 0.$

But $\sum v_n = \sum \frac{1}{2\sqrt{n}}$. is divergent. As p-series test $p = \frac{1}{2} < 1$.

Hence by comparison test the given series $\sum_{n=1}^{\infty} u_n$ is also divergent.

c) Here $u_n = \sin(1/n)$ also take $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty} \frac{\sin(1/n)}{\left(\frac{1}{n}\right)} = 1$

which is finite and non-zero.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. [as $\sum_{n=1}^{\infty}\frac{1}{n}$ is a harmonic series

Hence by comparison test (limit form), the series $\sum_{n=1}^{\infty} u_n$ is also divergent.